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TOWARD CALCULATION OF FREE-CONVECTIVE MOTION OF LIQUID OVER A LINEAR HEAT SOURCE

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The results of an analytical and numerical investigation of a laminar free-convective motion of liquid over a linear heat source are presented. The characteristic features of velocity and temperature field as functions of the density parameter and Prandtl number have been studied. A table of numerical solutions is given.

Introduction. Vertical free-convective jet flows originating over point or linear heat sources have a wide range of practical applications. Therefore, a great number of experimental and theoretical works have been devoted to their study. However, the investigations are mainly restricted to the case of linear dependence of density on temperature. Relatively scant attention has been given to analysis of the hydrodynamic and thermal characteristics of free-convective jets in the case of nonlinear dependence of density on temperature [1–5]. The available data do not give sufficient information on the specific features of the structure of flow fields and on the laws governing their development. At the present time, the mentioned range of problems has not been fully and systematically surveyed in the technical literature.

Below, we present the results of a complex analytical and numerical investigation of a fully laminar boundary layer in the Boussinesq approximation.

Basic Equations. We will consider a vertical plane fluid flow caused by a horizontal linear heat source of intensity Q_0. We will introduce a Cartesian coordinate system — x, y — with the corresponding velocity components u and v. The origin of the system coincides with the position of the source and the axis is directed vertically upward. The properties of the fluid are considered constant, except for the density $\rho = \rho_\infty[1 - \beta_q(\Delta T)^q]$. Then the basic equations which describe a stationary laminar jet vertical motion are written in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta_q(\Delta T)^q\,, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \quad u\frac{\partial \Delta T}{\partial x} + v\frac{\partial \Delta T}{\partial y} = \frac{\nu}{\mathrm{Pr}}\frac{\partial^2 \Delta T}{\partial y^2}\,. \tag{1}$$

Further, from the conditions of symmetry and locality of the flow the boundary conditions for the velocity and temperature fields follow:

$$y = 0: \quad v = \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = 0\,;$$
$$y \to \infty: \quad u \to 0\,, \quad T \to T_\infty\,. \tag{2}$$

The statement of the problem is completed by writing the integral law of heat-flux conservation:

$$Q_0 = \rho C_p \int_{-\infty}^{+\infty} u\Delta T dy\,, \tag{3}$$

which follows from the third equation of system (1). Now we will go over to dimensionless variables. As a function of stream, coordinate, and temperature, we will use the quantities

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$$\psi(x,\eta)=\left[g\beta_q\nu^2\left(\frac{Q_0}{\rho C_p}\right)^q\right]^{\frac{1}{4+q}} f(\eta)\,x^{\frac{3}{4+q}}\,,\quad \eta=\left[\frac{g\beta_q}{\nu^2}\left(\frac{Q_0}{\rho C_p\nu}\right)^q\right]^{\frac{1}{4+q}} yx^{-\frac{1+q}{4+q}}\,,$$

$$\Delta T=\left[\frac{Q_0^4\nu^2}{g\beta_q(\rho C_p\nu)^4}\right]^{\frac{1}{4+q}} h(\eta)\,x^{-\frac{3}{4+q}}\,. \tag{4}$$

Then, in a self-similar approximation, we obtain the following system of ordinary differential equations for determining the unknown functions $f(\eta)$ and $h(\eta)$:

$$f'''+\frac{3}{4+q}ff''-\frac{2-q}{4+q}f'^2+h^q=0\,,\quad \frac{1}{\mathrm{Pr}}h''+\frac{3}{4+q}(fh)'=0\,,\quad f(0)=f''(0)=h'(0)=0\,,$$

$$f'(\infty)=h(\infty)=0\,,\quad 2\int_0^\infty f'hd\eta=1\,. \tag{5}$$

An analytical solution of the system of equations (5) has not been found, and therefore in [1] numerical integration of (5) was done, with given Pr and parameter q. The main difficulty of system (5) is due to the interrelationship of nonlinear equations which reflect the main specific feature of the free-convective motion of a fluid in a jet, namely, the temperature distribution exerts its influence on the hydrodynamics and therefore the boundary-layer structure depends explicitly on the Pr number. Additional problems are associated with the parameter q. At the same time, in the theory of free-convective jets, of definite interest is the finding of solutions in terms of the quadratures of the known elementary or special functions, called construction of exact solutions, not only for qualitative investigation of the process of momentum and heat transfer and analysis of asymptotic properties, but also as an effective standard for estimating the error of approximate numerical methods. The thing is that in numerical integration the boundary-value problem given by Eq. (5) is usually transformed to the Cauchy problem. Generally such an approach leads to the necessity of varying initial data in order to conjugate the sought-for solution to the given boundary-value conditions. The procedure of varying the initial parameters over the a priori unknown range of their change leads to a sharp increase in the amount of iterative computations and may considerably decrease the effectiveness and accuracy of numerical solutions.

Results Obtained and Their Analysis. We will show that the problem, Eqs. (5), has an exact solution. We assume that

$$f=a\tanh b\eta\,, \tag{6}$$

where a and b are unknown coefficients. Integrating the second equation of system (5), we will obtain

$$h=c\,(1-\tanh^2 b\eta)^{\frac{3a\mathrm{Pr}}{2b(4+q)}} \tag{7}$$

(c is an integration constant). We note that functions (6) and (7) automatically satisfy the given boundary conditions. Then the first equation of system (5) acquires the form

$$-2ab^3(1-3z^2)-\frac{6a^2b^2}{4+q}z^2-\frac{2-q}{4+q}a^2b^2(1-z^2)+c^q(1-z^2)^{k-1}=0\,, \tag{8}$$

in which $z=\tanh b\eta$ and $k=\dfrac{3aq\mathrm{Pr}}{2b(4+q)}$ are introduced to save space. It can be easily seen that only two cases of the conversion of Eq. (8) into identity are possible: $k=1$ and $k=2$. In the first case,

$$a = 6b\,, \quad c = \left(24\,\frac{(5-q)}{4+q}\,b^4 \right)^{1/q},$$

in the second,

$$a = \frac{2}{3}\,(4+q)\,b\,, \quad c = \left(\frac{4}{9}\,(4+q)\,(5-q)\,b^4 \right)^{1/q}.$$

Consequently, the problem, Eqs. (5), admits two classes of exact solutions which have the form

$$1)\ \mathrm{Pr} = \frac{4+q}{9q}: \ f' = 6\alpha^2\,(1 - \tanh^2 \alpha\eta)\,, \quad h = \left(24\,\frac{5-q}{4+q}\,\alpha^4 \right)^{1/q} (1 - \tanh^2 \alpha\eta)^{1/q},$$

$$\alpha = \left[\frac{\Gamma\left(\frac{3}{2} + \frac{1}{q}\right)}{6\,\sqrt{\pi}\,\Gamma\left(1 + \frac{1}{q}\right)} \left(\frac{4+q}{24\,(5-q)} \right)^{1/q} \right]^{\frac{q}{4+q}};$$

$$2)\ \mathrm{Pr} = \frac{2}{q}: \ f' = \frac{2}{3}\,(4+q)\,\alpha^2\,(1 - \tanh^2 \alpha\eta)\,, \quad h = \left(\frac{4}{9}\,(4+q)\,(5-q)\,\alpha^4 \right)^{1/q} (1 - \tanh^2 \alpha\eta)^{2/q},$$

$$\alpha = \left[\frac{3\Gamma\left(\frac{3}{2} + \frac{2}{q}\right)}{2\,(4+q)\,\sqrt{\pi}\,\Gamma\left(1 + \frac{2}{q}\right)} \frac{1}{\left(\frac{4}{9}\,(4+q)\,(5-q) \right)^{1/q}} \right]^{\frac{q}{4+q}}. \tag{9}$$

We note that the values of the coefficient α in Eqs. (9) are determined by normalization of Eqs. (5). Thereafter, the earlier-found analytical solutions at fixed values $q = 1$ [6, 7] and $q = 2$ [8] are obtained as a particular case from the results of the present work. The relations

$$\frac{ux}{\nu} = f'(\eta)\,\mathrm{Gr}_x^{\frac{2}{4+q}}, \ \Delta T = \frac{Q_0}{\mu C_p}\,h(\eta)\,\mathrm{Gr}_x^{-\frac{1}{4+q}}, \ \frac{m}{\mu} = 2f(\infty)\,\mathrm{Gr}_x^{\frac{1}{4+q}}, \ \eta = \frac{y}{x}\,\mathrm{Gr}_x^{\frac{1}{4+q}}, \ \mathrm{Gr}_x = \frac{g\beta_q}{\nu^2}\left(\frac{Q_0}{\mu C_p}\right)^q x^3 \tag{10}$$

in combination with the analytical expressions for the functions f' and h, allow one to determine all of the basic parameters of the jet vertical free-convective motion of a fluid over the main (self-similar) section. Due to the interaction of the fields of velocity and temperature, of particular interest is the elucidation of the role of Pr number in the formation of flow structure. Since the region of application of Eqs. (9) is limited by the equations of coupling between q and Pr at which the problem posed admits integration in quadratures, numerical solution of system (5) was also performed by adjusting it to the corresponding Cauchy problem. The relationship of the Cauchy problem that requires the knowledge of two additional parameters, $f'(0)$ and $h(0)$, with the initial boundary-value one was made by integrating the system of ordinary differential equations to find the unknown functions $f(\eta)$ and $h(\eta)$ by a fourth-order accuracy Hamming method with automatic selection of a step up to such a value of $\eta = \eta_\infty$ (η_∞ is the numerical approximation of the mathematical point $\eta = \infty$) at which asymptotic boundary conditions are satisfied. As the initial parameters were verified, the value of η_∞ was increased to eliminate its influence on the sought-for results. The computations were considered completed if the values of the functions f' and h at "infinity" were approximately equal to 10^{-10} and the function f strived asymptotically to a constant when $\eta \to \eta_\infty$. The numerical analysis was carried out with "open eyes"

TABLE 1. Results of Numerical Calculation of a Plane Free-Convective Jet

Pr	$f'(0, \mathrm{Pr})$	$h(0, \mathrm{Pr})$	$f(\infty, \mathrm{Pr})$
	$q = 1$		
0.01	0.5274227	0.0720834	9.7053784
0.03	0.6177124	0.1114223	6.2810981
0.1	0.7103305	0.1784027	3.9402534
0.3	0.7754418	0.2713389	2.6540461
5/9	0.8009304	0.3421277	2.1921638
0.7	0.8087151	0.3732810	2.0589670
0.72	0.8096107	0.3772794	2.0440952
1	0.8193698	0.4275389	1.8909634
5	0.8596752	0.8225673	1.5186558
6.7	0.8664069	0.9346655	1.4893659
7	0.8673922	0.9529083	1.4855247
10	0.8751567	1.1174208	1.4585513
100	0.9109417	3.3017885	1.3822071
1000	0.9258191	10.2170496	1.3636586
	$q = 4/3$		
4/9	0.7139613	0.3360455	2.0697266
1.5	0.7721564	0.5122860	1.6569385
5	0.8379172	0.8125824	1.5252395
7	0.8585357	0.9325449	1.5119974
10	0.8811922	1.0827628	1.5044703
11.4	0.8896887	1.1447140	1.5030612
100	1.0390366	2.9959666	1.5399571
	$q = 1.5$		
4/3	0.7407311	0.4958508	1.6480334
7	0.8553082	0.9216304	1.5304763
10	0.8848807	1.0653317	1.5316236
11.4	0.8961037	1.1244352	1.5333437
100	1.1051979	2.8610482	1.6224878
	$q = 1.894816$		
7	0.8487920	0.8955079	1.5820451
10	0.8934681	1.0253988	1.6022126
11.4	0.9107464	1.0785093	1.6109321
11.6	0.9130744	1.0857922	1.6121378
100	1.2643872	2.5835038	1.8254563
	$q = 2$		
1/3	0.5873677	0.3391169	1.8772870
1	0.6551853	0.4632860	1.6188704
6.7	0.8415191	0.8743897	1.5944646
7	0.8471856	0.8886093	1.5971306
10	0.8956044	1.0151754	1.6221215
11.4	0.9144049	1.0668515	1.6326421
100	1.3070689	2.5182830	1.8811542

with the aid of analytical dependences (9) as a ground to test the accuracy of the solutions constructed and to discern the symptoms of their numerical "ailment." The results of the calculations are presented in Table 1. We will note some of the specific features of the results obtained. As is seen from the formulas

$$u \sim Q_0^{\frac{2q}{4+q}}, \quad \Delta T \sim Q_0^{\frac{4}{4+q}}, \quad m \sim Q_0^{\frac{q}{4+q}}, \tag{11}$$

the basic parameters of the jet flow depend differently on the heat-source power, with the excess temperature being most sensitive to a change in Q_0. As to the laws governing a change in velocity, temperature, and mass flow rate per second with distance from the linear horizontal heat source, here $u \sim x^{\frac{2-q}{4+q}}$ and $m \sim x^{\frac{3}{4+q}}$ increase, whereas $\Delta T \sim x^{-\frac{3}{4+q}}$ decreases. Attention is drawn to the independence of u at $q = 2$ of the distance up to the heat source (a fact similar to the independence of the axial velocity in an axisymmetrical free-convective jet [9]). Based on the numerical data obtained, a conclusion can be drawn that the values of $f'(0)$ and $h(0)$ turn out to be higher for fluids with a high Prandtl number. An important feature of the distribution of $f(\infty)$ is its dependence on the parameter q. Thus, when $1 < q \le 2$ the quantity $f(\infty)$ decreases to a certain minimum value with an increase in the Prandtl number, whereas with a further increase in Pr its growth is observed. At the same time, for the case $q = 1$, $f(\infty)$ decreases monotonically in the entire range of Pr numbers considered. But if we prescribe a constant Prandtl number, then on increase of the parameter q from 1 to 2 the value of $f(\infty)$ increases and that of $h(0)$ decreases, with the rate of change being the higher, the higher the Prandtl number.

And finally, analyzing Table 1, we see that at a constant Prandtl number the value of $f'(0)$ can decrease (Pr = 7) or increase (Pr = 10, 100) with an increase in q.

Conclusions. The analytical and numerical solutions presented allow us to gain a deeper understanding of the laws governing a steady-state, free-convective motion of a fluid over a linear heat source and to predict the influence of various factors on a change in the hydrodynamic and thermal characteristics of a jet flow, as well as provide new information needed for engineering calculations, and can also be useful for designing.

NOTATION

C_p, heat capacity at constant pressure, J/(kg·K); g, free fall acceleration, m/sec^2; Gr_x, local Grashof number; m, mass flow rate per second, kg/(m·sec); Pr, Prandtl number; Q_0, flux of excess heat capacity, J/(m·sec); q, density parameter; T, temperature, K; u and v, longitudinal and transverse velocity components, m/sec; x and y, longitudinal and transverse coordinates, m; β_q, temperature coefficient, 1/K^q; Γ, gamma-function; $\Delta T = T - T_\infty$, excess temperature, K; μ, coefficient of dynamic viscosity, kg/(m·sec); ν, coefficient of kinematic viscosity, m^2/sec; ρ, density, kg/m^3. Subscripts: ∞, surrounding fluid.

REFERENCES

1. J. C. Mollendorf, R. S. Johnson, and B. Gebhart, Several plume flows in pure and saline water at its density extremum, *J. Fluid Mech.*, **113**, 269–282 (1981).
2. Y. Joshi and B. Gebhart, An experimental study of plane plumes in cold water, *Trans ASME. J. Heat Transfer*, **105**, No. 2, 248–254 (1983).
3. Y. Joshi and B. Gebhart, Vertical transient natural convection flows in cold water, *Int. J. Heat Mass Transfer*, **27**, No. 9, 1573–1582 (1984).
4. O. G. Martynenko, V. N. Korovkin, and Yu. A. Sokovishin, A laminar swirled free-convection jet, *Int. J. Heat Mass Transfer*, **28**, No. 2, 371–382 (1985).
5. V. N. Korovkin and A. P. Andrievskii, Free convection with a nonlinear dependence of density on temperature: plane problems, *Inzh.-Fiz. Zh.*, **73**, No. 2, 381–386 (2000).

6. L. J. Crane, Thermal convection from a horizontal wire, *ZAMP*, **10**, No. 5, 453–460 (1959).
7. T. Fujii, Theory of the steady laminar natural convection above a horizontal line heat source and point heat source, *Int. J. Heat Mass Transfer*, **6**, No. 7, 597–606 (1963).
8. V. N. Korovkin and S. L. Semin, Mixed convection in laminar vertical jets, *Vestsi Akad. Navuk BSSR*, *Ser. Fiz.-Energ. Navuk*, No. 2, 77–83 (1986).
9. O. G. Martynenko, V. N. Korovkin, and Yu. A. Sokovishin, *Theory of Buoyant Jets and Wakes* [in Russian], Nauka i Tekhnika, Minsk (1991).